PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 9, Fasc. 2 (1988), p. 45–49

# A KRONECKER-PRODUCT DESIGN AND ITS REDUCIBLE ASSOCIATE CLASSES

BY

S. KAGEYAMA (HIROSHIMA) AND R. N. MOHAN (NUZVID)

*Abstract.* Using incidence matrices $N_i$ of balanced incomplete block (BIB) designs and their complementary incidence matrices $N_i^*$ for $i = 1, 2$, a partially balanced incomplete block (PBIB) design in the form $N_1 \otimes N_2 + N_1^* \otimes N_2 + N_1 \otimes N_2^*$ is dealt with. Necessary and sufficient conditions for this rectangular PBIB design to be reducible to 2-associate PBIB designs are discussed. It is also shown that this type of designs is not reducible to any group divisible PBIB design.

**1. Introduction.** The Kronecker product of designs and reduced designs were defined by Vartak [7], who did not consider the association scheme concerning these designs explicitly. Reduced designs, together with association schemes matching these designs, were discussed by Kageyama [3, 4, 5]. It should also be remarked that an association scheme can be defined and characterized independently of treatment-block incidence of the design (cf. [1]). When the coincidence numbers (i.e., $\lambda_i$-parameters) of a PBIB design are not all different, some associate classes of the PBIB design, based on a certain association scheme, may not be all distinct. In this case, there is only a possibility of reducing the number of associate classes of the PBIB design.

The idea in this paper is similar to that in [3] and [4]. But we can derive new results by presenting necessary and sufficient conditions for a PBIB design with at most three associate classes constructed by the Kronecker product of two BIB designs, to be reducible to PBIB designs with only two distinct associate classes. It is also shown that the design considered here is not reducible to a group divisible PBIB design. This may be an interesting case regarding the reduction of associate classes in a rectangular PBIB design.

The notation used here are coincident with those generally used. The definitions of BIB designs, PBIB designs and association schemes can be found in [1], [2] and [6].

**2. Kronecker product of two BIB designs.** Let $N_i$ be a BIB design with parameters $v_i, b_i, r_i, k_i, \lambda_i$, and $N_i^*$ be its complementary BIB design with parameters $v_i^* = v_i$, $b_i^* = b_i$, $r_i^* = b_i - r_i$, $k_i^* = v_i - k_i$, $\lambda_i^* = b_i - 2r_i + \lambda_i$ for $i = 1, 2$. Consider the Kronecker product of these designs of the form $N = N_1 \otimes N_2 + N_1^* \otimes N_2 + N_1 \otimes N_2^*$, which is also equivalent, as a matrix, to $J_{v_1 \times b_1} \otimes N_2 + N_1 \otimes N_2^*$ or $N_1 \otimes J_{v_2 \times b_2} + N_1^* \otimes N_2$, having different coincidence numbers, where $J_{s \times t}$ is the $s \times t$ matrix of 1's, and especially $G_s = J_{s \times s}$. But, here, we consider designs with incidence block structure in the form of $N$ as above only.

From the properties of the BIB designs it follows that, for $i = 1, 2$,

$$
\begin{aligned}
&N_i N_i' = (r_i - \lambda_i) I_{v_i} + \lambda_i G_{v_i},\\
(2.1)\qquad &N_i^* N_i^{*\prime} = (r_i - \lambda_i) I_{v_i} + \lambda_i^* G_{v_i},\\
&N_i N_i^{*\prime} = (\lambda_i - r_i) I_{v_i} + (r_i - \lambda_i) G_{v_i},
\end{aligned}
$$

where $I_{v_i}$ is the identity matrix of order $v_i$. Since these matrices (2.1) are symmetrical and mutually commutative, there exist orthogonal matrices $P_i$ which make all of them diagonal simultaneously, such that

$$
\begin{aligned}
&P_i N_i N_i' P_i' = D_i = \operatorname{diag}\{r_i k_i, r_i - \lambda_i, \ldots, r_i - \lambda_i\},\\
(2.2)\qquad &P_i N_i^* N_i^{*\prime} P_i' = D_i^* = \operatorname{diag}\{(b_i - r_i)(v_i - k_i), r_i - \lambda_i, \ldots, r_i - \lambda_i\},\\
&P_i N_i N_i^{*\prime} P_i' = \tilde{D}_i = \operatorname{diag}\{(v_i - 1)(r_i - \lambda_i), \lambda_i - r_i, \ldots, \lambda_i - r_i\},
\end{aligned}
$$

where $\operatorname{diag}\{a_1, \ldots, a_n\}$ denotes a diagonal matrix with diagonal elements $a_1, \ldots, a_n$, here $n$ being $v_i$, respectively, for $i = 1, 2$. Hence it follows, from (2.2), that

$$
\begin{aligned}
(2.3)\qquad (P_1 \otimes P_2) NN' (P_1 \otimes P_2)' = D_1 \otimes D_2 + D_1 \otimes D_2^* + D_1^* \otimes D_2 +\\
+ 2(D_1 \otimes \tilde{D}_2 + \tilde{D}_1 \otimes D_2 + \tilde{D}_1 \otimes \tilde{D}_2).
\end{aligned}
$$

From the structure of $N$, the parameters of the design can be expressed as follows:

$$
\begin{aligned}
&v' = v_1 v_2,\ b' = b_1 b_2,\ r' = r_1 r_2 + r_1 (b_2 - r_2) + r_2 (b_1 - r_1),\\
&k' = k_1 k_2 + k_1 (v_2 - k_2) + k_2 (v_1 - k_1),\\
&\lambda_1' = r_1 \lambda_2 + (b_1 - r_1) \lambda_2 + r_1 (b_2 - 2r_2 + \lambda_2),\\
&\lambda_2' = \lambda_1 r_2 + (b_1 - 2r_1 + \lambda_1) r_2 + \lambda_1 (b_2 - r_2),\\
&\lambda_3' = \lambda_1 \lambda_2 + (b_1 - 2r_1 + \lambda_1) \lambda_2 + \lambda_1 (b_2 - 2r_2 + \lambda_2).
\end{aligned}
$$

Furthermore, it follows from (2.2) and (2.3) that the eigenvalues of $NN'$ and their multiplicities are as follows:

$\varrho'_0 = r'k'$ with multiplicity 1,

$\varrho'_1 = (r_2-\lambda_2)(b_1-r_1)(v_1-k_1)$ with multiplicity $v_2-1$,

$\varrho'_2 = (r_1-\lambda_1)(b_2-r_2)(v_2-k_2)$ with multiplicity $v_1-1$,

$\varrho'_3 = (r_1-\lambda_1)(r_2-\lambda_2)$ with multiplicity $(v_1-1)(v_2-1)$.

The same procedure as in [3], [4] and [5] shows that the design $N = N_1 \otimes N_2 + N_1^* \otimes N_2 + N_1 \otimes N_2^*$ is a PBIB design with at most three associate classes. Then among the $v_1 v_2$ treatments in the design $N$ a rectangular association scheme can naturally be defined as follows. Let there be $v = v_1 v_2$ treatments arranged in a rectangle of $v_1$ rows and $v_2$ columns. With respect to each treatment, the first associates are the other $n_1 = v_2-1$ treatments of the same row, the second associates are the other $n_2 = v_1-1$ treatments of the same column, and the remaining $n_3 = (v_1-1)(v_2-1)$ treatments are the third associates ([6], Section 8.12.1).

In order to have a two-class association scheme, the following relations can first be found:

$$\lambda'_1 = \lambda'_2 \text{ iff } (r_1-\lambda_1)b_2-(r_2-\lambda_2)b_1 = r_2\lambda_1-r_1\lambda_2,$$

$$\lambda'_1 = \lambda'_3 \text{ iff } b_2 = 2r_2-3\lambda_2,$$

(2.4)
$$\lambda'_2 = \lambda'_3 \text{ iff } b_1 = 2r_1-3\lambda_1.$$

$$\varrho'_1 = \varrho'_2 \text{ iff } (r_2-\lambda_2)(b_1-r_1)(v_1-k_1) = (r_1-\lambda_1)(b_2-r_2)(v_2-k_2).$$

$$\varrho'_1 = \varrho'_3 \text{ iff } b_1 = 2r_1-\lambda_1,$$

$$\varrho'_2 = \varrho'_3 \text{ iff } b_2 = 2r_2-\lambda_2.$$

However, since the coincidence number of a complementary design of a BIB design satisfying $b_i = 2r_i-3\lambda_i$ is $b_i-2r_i+\lambda_i = -2\lambda_i$, which is negative, there does not exist a BIB design with $b_i = 2r_i-3\lambda_i$ for $i = 1, 2$. This, with (2.4), implies that relations $\lambda'_1 \neq \lambda'_3$ and $\lambda'_2 \neq \lambda'_3$ hold in this design $N$. Hence, from the definition of the group divisible association scheme ([6], Section 8.4), the following result concerning the reduction of associate classes is established:

PROPOSITION A. *The above* 3-*associate rectangular PBIB design* $N_1 \otimes N_2 + N_1^* \otimes N_2 + N_1 \otimes N_2^*$ *is not reducible to any* 2-*associate group divisible PBIB design.*

Note that a BIB design satisfying $b_i = 2r_i-\lambda_i$ is just the design $G_{v_i}-I_{v_i}$ or a juxtaposition of several copies of it.

Now, (2.4) and Proposition A imply that if the PBIB design $N$ with three associate classes is a PBIB design with at most two distinct associate classes, then only the combination of the first and the second associate classes as a new associate class can be considered (with the third associate class being invariant as a new second associate class) to obtain a possible reduction to an $L_2$ association scheme. In the present case, from the definition of the $L_2$ association scheme ([6], Section 8.4), we must have a condition $v_1 = v_2$. Thus, from the structure of the association schemes and (2.4), we can obtain necessary conditions for the reduction as follows:

$$v_1 = v_2 \quad \text{and} \quad (r_1-\lambda_1)b_2-(r_2-\lambda_2)b_1 = r_2\lambda_1-r_1\lambda_2, \tag{2.5}$$

$$(r_2-\lambda_2)(b_1-r_1)(v_1-k_1) = (r_1-\lambda_1)(b_2-r_2)(v_2-k_2). \tag{2.6}$$

Conversely, if the parameters of a PBIB design $N_1 \otimes N_2+N_1^* \otimes N_2 +N_1 \otimes N_2^*$ satisfy (2.5), then the design can obviously lead to a PBIB design with at most two distinct associate classes. Among the $v^2$ treatments in the reduced design, where $v = v_1 = v_2$, an $L_2$ association scheme can be formed under (2.5), by combining the first associate class and the second associate class. In this case, from a property of an association algebra of the $L_2$ association scheme, $\varrho_1' = \varrho_2'$ (i.e., (2.6)) must hold necessarily.

Thus, the following has been obtained:

THEOREM A. *Given two BIB designs, $N_i$, with parameters $v_i$, $b_i$, $r_i$, $k_i$, $\lambda_i$ for $i = 1, 2$, a PBIB design $N_1 \otimes N_2+N_1^* \otimes N_2+N_1 \otimes N_2^*$, of the rectangular association scheme with at most three associate classes, is reducible to a PBIB design of the $L_2$ association scheme, with only two distinct associate classes, iff*

$$v_1 = v_2 \quad \textit{and} \quad (r_1-\lambda_1)b_2-(r_2-\lambda_2)b_1 = r_2\lambda_1-r_1\lambda_2.$$

Remark. This theorem is different from Theorem 4 of Kageyama [3]. An example of Theorem A can always be produced by taking $N_1 = N_2$ as starting BIB designs.

Since in BIB designs the equalities $v_1 = v_2$ and $k_1 = k_2$ yield a relation $r_1\lambda_2 = r_2\lambda_1$, it follows that $(r_1-\lambda_1)b_2 = (r_2-\lambda_2)b_1$ is equivalent to (2.5) [the second condition] and to (2.6), if the above equalities hold. This observation together with Theorem A gives also the following

COROLLARY A. *Given two BIB designs, $N_i$, with parameters $v$, $b_i$, $r_i$, $k$ and $\lambda_i$ for $i = 1, 2$, a necessary and sufficient condition for a PBIB design $N_1 \otimes N_2 +N_1^* \otimes N_2+N_1 \otimes N_2^*$, of a rectangular association scheme with at most three associate classes, to be reducible to a PBIB design of the $L_2$ association scheme with two distinct associate classes, is that $(r_1-\lambda_1)b_2 = (r_2-\lambda_2)b_1$.*

Remark. For some kind of similarity of an expression of necessary and sufficient conditions for the reduction, refer to Theorem 4 in [3] and to [4].

**3. Additional remark.** We can consider various other forms as Kronecker products of some incidence matrices of BIB designs, such as $N_1 \otimes N_2 \otimes N_3 + N_1^* \otimes N_2 \otimes N_3 + N_1 \otimes N_2^* \otimes N_3 + N_1 \otimes N_2 \otimes N_3^*$, $N_1 \otimes N_2 \otimes N_3 + N_1^* \otimes N_2 \otimes N_3 + N_1 \otimes N_2^* \otimes N_3^*$, and their generalizations. But, these cases may also be treated as a routine application of the present approach with some modification. Hence they are omitted here.

**Acknowledgements.** The authors thank the referee for his thorough and constructive comments which improved the presentation. The first author was supported in part by Grant 321-6066-58530013 (Japan) and Polish Academy of Sciences, while the second author was supported in part by research grant code no. 12355, Dt.27/2/82 and no. 1607, Dt.29/10/85 of University Grants Commission, New Delhi, India.

## REFERENCES

[1] R. C. Bose and D. M. Mesner, *On linear associative algebra corresponding to association schemes of partially balanced designs*, Ann. Math. Statist. 30 (1959), p. 21–38.

[2] W. H. Clatworthy, *Tables of Two-Associate-Class Partially Balanced Designs*, NBS Applied Mathematics Series 63, U.S. Department of Commerce, Washington 1973.

[3] S. Kageyama, *On the reduction of associate classes for certain PBIB designs*, Ann. Math. Statist. 43 (1972), p. 1528–1540.

[4] – *On the reduction of associate classes for the PBIB design of a certain generalized type*, Ann. Statist. 2 (1974), p. 1346–1350.

[5] – *Reduction of associate classes for block designs and related combinatorial arrangements*, Hiroshima Math. J. 4 (1974), p. 527–613.

[6] D. Raghavarao, *Constructions and Combinatorial Problems in Design of Experiments*, John Wiley, New York 1971.

[7] M. N. Vartak, *On an application of Kronecker product of matrices to statistical designs*, Ann. Math. Statist. 26 (1955), p. 420–438.

Department of Mathematics
Faculty of School Education
Hiroshima University
Shinonome, Hiroshima 734, Japan

Department of Mathematics
D. A. R. College
Nuzvid A. P.
India

*Received on 11. 11. 1986;*
*revised version on 30. 3. 1987*